2924. Proposed by FLORENCE P. LEWIS, Goucher College.

Given a triangle and a conic. Through each vertex of the triangle there pass two lines harmonic to the tangents through that point and to the sides of the triangle. Prove that the six lines so found pass by threes through four points.

2925. Proposed by F. V. MORLEY, New College, Oxford, Eng.

A regular polygon of $2n+1$ sides will have only $n-1$ diagonals of different lengths (*e.g.*, the regular heptagon has two distinct diagonals). Call the side of such a polygon a_1, and the $n-1$ diagonals in order of size $a_2 \cdots a_n$. Then if the circumscribed circle has radius unity, $\sum_{i=1}^{n} a_i^2 = 2n+1$; in words, the sum of the squares of the distinct lengths obtained by joining an odd number of regularly spaced points on a unit circle is equal to the number of such points.

2926. Proposed by T. M. SIMPSON, Randolph-Macon College, Ashland, Va.

Solve the differential equation,

$$(y + x^2)dx + (x - x^2y)dy = 0.$$

2927. Proposed by PHILIP FRANKLIN, Princeton University.

Prove that the only positive integral values greater than unity which satisfy the equation $3^x - 2^y = \pm 1$ are $x = 2$, $y = 3$. (Cf. Carmichael, *Diophantine Analysis*, 1915, p. 116, exercise 69.)

NOTES.

22. Huge Numbers.—"What is the largest number that we can express by three digits?" The answer is $N = 9^{(9^9)}$, that is $9^{387,420,489}$. C. A. Laisant drew attention to this number in his *Initiation Mathématique*, Paris, 1906 (English edition, London, 1913). He there remarks that in decimal numeration this number would have 369,693,100 figures. To write it on a single strip of paper, supposing that each figure occupied a space of one fifth of an inch, the length of the strip would need to be 1,166 miles, 1,690 yards, 1 foot, 8 inches. In this connection C. E. Guillaume remarked (*Revue Générale des Sciences*, vol. 17, 1906, p. 878) that under the same conditions to write $10^{(10^{10})}$, we would need a strip of paper long enough to encircle the earth.

Writing in May, 1913, A. C. D. Crommelin stated (*Journal of the British Astronomical Association*, vol. 23, pp. 380–381) that he had come across the problem with which this note commences "in an old logarithm book." By the aid of 61-figure logarithms of certain numbers given in Hutton's tables Dr. Crommelin found $\log N = 369{,}693{,}099.6315703587\cdots$; whence the number of figures indicated above. He found the first 28 of the figures to be 428,124,773,175,747,048,036,987,115,9 and the last three to be 289.

"A knowledge of 30 figures out of 300 million," he continued, "may seem trifling, but in reality the error involved in taking all the remaining figures as zeros is only one part in a thousand quadrillions. If the number were printed with 16 figures to an inch (about the tightest packing for decent legibility), it would extend over 364.7 miles. . . . If printed in a series of large volumes we might get 14,000 figures to a page, and with 800 pages to the volume it would fill 33 volumes. There are more than twice as many digits in the number as there are letters in the whole of the *Encyclopædia Britannica*.

"To find the largest number suggested by sideral astronomy I took the following. Both Very and See have expressed the opinion that certain visible objects may be at a distance of a million light years; I imagine a solid sphere of platinum of this radius, and find how many electrons it contains. From Duncan's *The New Knowledge*, p. 65, I find that the log of the number of electrons in a cubic centimetre of water is 16.469. Taking the density of platinum as 21.5 the log of the number of electrons in a cubic inch of it is 19.016, and the log of the volume of the

huge sphere in cubic inches is 71.3362. Whence the log of the number of electrons it contains is 90.352, and the corresponding number is 225 followed by 88 zeros. At 16 figures to the inch this would take $5\frac{3}{4}$ inches. . . .

"To find the radius of a sphere of platinum that would contain 9^{9^9} electrons, we must multiply our million-light-year radius by a number whose log is 123,231,003.093, *i.e.*, the multiplier is 1239 followed by 123,231,000 zeros. In fact, that gigantic sphere would exceed the million-light-year sphere in a far higher ratio than that exceeds the size of one electron.[1] Hence we may take it as morally certain that we can write with three digits a number vastly exceeding the number of electrons in the whole of creation, which is a somewhat startling fact. Indeed, even the number 4^{4^4} (which is 13407813 followed by 147 other figures) probably exceeds the number of electrons in creation. At least it equals the number of electrons in a solid platinum sphere that exceeds the million-light-year sphere in the same proportion that that exceeds a sphere 206 inches in radius."

In May, 1915, D. G. McIntyre considered (*Journal of the British Astronomical Association*, vol. 26, pp. 46–47) the "rather pretty" problem of determining the last figures of N, three of which, 289, were given by Dr. Crommelin. He found the last eight figures to be 17177289.

In *The Observatory* for July, 1920, H. H. Turner refers to N and to his choice, as rather simpler for expression and not very different in magnitude, of the number $10^{(10^{10})}$ which, when written out fully, is unity followed by ten thousand million zeros. "The validity of the substitution remained unchallenged until the other day, when Dr. Crommelin realised that injustice had been done. He accused me of treating as comparable, let us say, a billion times the distance of Dr. Shapley's furthest star cluster and a wave-length of light; for the ratio of these would, he said, be no greater than that of $10^{10^{10}}$ to 9^{9^9}. The accusation could not be repelled, and I tendered apologies as gracefully as the magnitude of the error would allow. But the accuser had apparently not reaped his advantage to the full. A few days later I received the following post card from him:

"I greatly understated the ratio of $10^{10^{10}}$ to 9^{9^9}, which is a number of some 900 million figures, and would be several thousands of miles long if written out at 16 figures to the inch. On the other hand, the ratio I mentioned yesterday (viz., a billion times the distance of Shapley's furthest cluster to the billionth of an inch) would have something under 50 figures in it, and could be written in the width of this postcard.

"I fear the framing of a suitable acknowledgment is beyond me."

Finally, in *Journal of the British Astronomical Association*, April, 1921, J. W. Meares comments on $9!^{(9!^{9!})}$ and finds that the value of his number is greater than 10 to the power $10^{2000000}$ but less than 10 to the power $10^{2000001}$. On this Dr. Crommelin commented: "If one allows the introduction of algebraic symbols the number $\infty^{\infty^{\infty}}$ has some claims on our attention. Perhaps I may be allowed to quote an old college rhyme:

"'There was a professor of Trinity
Who found the square root of infinity;
But in counting the digits
He was seized with the fidgets,
Dropped Science and took to Divinity.'"

One wonders if reference is here made to George Salmon, of Trinity College, Dublin, whose classic mathematical works have throughout the world been the delight of generations, but whose works on "Divinity," written after he had "dropped" mathematics, are known to few! ARC.

[1] "Indeed we should have to carry out enlargement in the ratio of a million-light-year sphere to an electron more than a million times in succession before we get a sphere of the size enlarged."

23. The Wallace Line and Wallace Point, and Some Generalizations.—In 1799, William Wallace[1] stated the theorem: Given a circle and a triangle inscribed in it, the feet of the perpendiculars on the sides of the triangle from any point of the circle are collinear.[2] Such a line is the Wallace line of the point. The first generalization of the theorem was in 1822 by Poncelet[3] who showed that the perpendiculars on the sides of the triangle may be replaced by obliques making, in cyclic order, equal angles with the sides. Both Wallace and Poncelet seem to have arrived at their results[4] by considering properties of the parabola: (*a*) the circumscribed circle of a triangle tangent to a parabola passes through its focus;[5] (*b*) the feet of the perpendiculars from a focus of a parabola on its tangents lie on the tangent at the vertex of the parabola.

As an application of his theorem, Wallace considered the question[6] of describing a parabola tangent to four given straight lines; he remarks that the focus of the parabola is determined by finding the second point of intersection of the circles circumscribing any two of the four triangles formed by the four given lines. In other words the four circles meet in a point—the Wallace point.[7] Without reference to the parabola, Wallace formulated the theorem again in 1804. It appeared also in Bland's *Geometrical Problems*, Cambridge, 1819, p. 259, before

[1] *The Mathematical Repository*, March, 1799, p. 111.

[2] This same theorem was given by Servois in February, 1814 (*Annales de Mathématiques* (Gergonne), vol. 4, p. 251). In a footnote he states the theorem in another form: "Through a point on the circumference of a circle three chords are drawn. Circles are described on these chords as diameters. The second points of intersection of these circles, taken in pairs, are collinear."

[3] *Traité des Propriétés Projectives*, 1822, p. 270.

[4] *The Mathematical Repository*, September, 1798, p. 81; Poncelet, *l.c.*, pp. 218–219.

[5] Lambert, *Insigniores Orbitae Cometorum Proprietates*, 1761, p. 5.

[6] *The Mathematical Repository*, March, 1799, p. 81.

[7] The centers of the four circles lie on a circle through the Wallace point—a result stated without proof by Steiner in 1828, *Annales de Mathématiques* (Gergonne), vol. 18, p. 302. From the above it is clear that there is scant justification for calling the Wallace point, the Miquel point, as has been done by S. Kantor (*Comptes rendus de l'Académie des Sciences de Vienne*, 1878), since Miquel derived the result in the periodical *Le Géomètre* (founded by Guillard), 1836, and later in Liouville's *Journal*, vol. 3, 1838, p. 486. So also J. L. Coolidge refers (*A Treatise on the Circle and the Sphere*, 1916, p. 87) to "the Miquel point of the four lines."

In 1871 W. K. Clifford remarked [in his *Common Sense of the Exact Sciences*, 1885, pp. 80–81] that Wallace's theorem concerning concurrent circles is the third of a series:

"If we take any two straight lines they determine a point, viz., their point of intersection.

"If we take three straight lines we get three such points of intersection; and these three determine a circle, viz., the circle circumscribing the triangle formed by the three lines.

"Four straight lines determine four sets of three lines by leaving out each in turn; and the four circles belonging to these sets of three meet in a point.

"In the same way five lines determine five sets of four, and each of these sets of four gives rise, by the proposition just proved, to a point. It has been shown by Miquel [*l.c.*], that these five points lie on the same circle.

"And this series of theorems has been shown [W. K. Clifford, 'Synthetic proof of Miquel's theorem,' *Oxford, Cambridge and Dublin Messenger of Mathematics*, vol. 5, 1871, p. 124] to be endless. Six straight lines determine six sets of five by leaving them out one by one. Each set of five has, by Miquel's theorem, a circle belonging to it. These six circles meet in the same point, and so on forever. Any even number ($2n$) of straight lines determines a point as the intersection of the same number of circles. If we take one line more, this odd number ($2n + 1$) determines as many sets of $2n$ lines, and to each of these sets belongs a point; these $2n + 1$ points lie on a circle."

it was given by Steiner in 1828 (compare problem 2898 of this MONTHLY, *1921*, 228).

Professor C. N. Mills of Tiffin, Ohio, suggested the following problem: "Given any triangle cut by a transversal through a fixed point on the base produced; through the fixed point and the intersections of the transversal with each side and the adjacent vertex at the base circles are drawn. Show that the locus of the intersection of these two circles is the circumcircle of the given triangle." From what has been given above it is clear that the point in which the circles meet is the Wallace point for the particular position of the transversal, which is the Poncelet line for that Wallace point.

Another generalization of Wallace's theorem stated without proof [1] (anonymously) in July, 1823, was as follows: (*a*) If from any point of a circle concentric with the circumscribed circle of a triangle, perpendiculars are dropped on the three sides, the area of the triangle, whose vertices are the feet of the perpendiculars, is constant. When this circle becomes the circumscribed circle the area vanishes. (*b*) If two circles concentric with the circumscribed circle are such that the sum of the squares of their radii is equal to twice the square of the radius of the circumscribed circle the two triangles formed as above are equivalent.

In 1870 Combette gave among other results: [2] (*a*) If P be an assumed point, and D, E, F its projections on the triangle ABC, the locus of P, when the area of the triangle DEF is constant, is the circumference of a circle concentric with the circumscribed circle of ABC; (*b*) For every value of the area of DEF included between zero and one fourth of ABC, the locus of P will consist of two circumferences concentric with the circumscribed circle, the one interior and the other exterior to it; (*c*) The sums of the squares of the radii of these two circumferences will be double the square of the circumscribed radius; (*d*) The locus of P does not change its nature or its center when the perpendiculars let fall on the sides become lines all making equal angles with the sides; (*e*) If the triangle ABC is replaced by a plane polygon, and the projections on its sides of a point P in the plane are joined in order, the locus of P when this area is constant is still a circle which has always the same center whatever be the value of the area;[3] (*f*) When the point P is taken in space and projected on the sides of a plane polygon, the locus of P, when the area obtained by joining the projections is constant, becomes a cylindrical surface of revolution whose axis, perpendicular to the plane of the polygon, is always the same for all values of the area.

The locus of the points whose projections on the planes of the faces of a

[1] *Annales de Mathématiques* (Gergonne), vol. 14, p. 28; Gergonne was probably the author. Analytic proofs are given on pages 280–293, the latter being by the great Sturm.

[2] *Revue des Sociétés Savantes*, vol. 5, pp. 203–233; compare J. S. Mackay, *Proceedings of the Edinburgh Mathematical Society*, vol. 9, pp. 86–87.

[3] This result was first enunciated for a *regular* polygon, without proof, by L'Huilier in *Bibliothèque Universelle*, March, 1824, p. 169. Proofs were given in *Annales de Mathématiques* (Gergonne), vol. 15, by "abonné," July, 1824, pp. 45–55; and by Sturm, February, 1825, pp. 250–252. This was generalized as above (and with further interesting results) by Steiner, in *Journal für die reine und angewandte Mathematik*, vol. 1, 1826, pp. 51–52; see also vol. 2, 1827, p. 265, and a further generalization (p. 263).

tetrahedron are coplanar is a cubic surface, S, through the edges of the tetrahedron and having the vertices of the tetrahedron as nodes.[1]—The surfaces S, for the 15 tetrahedra determined by any six planes, meet in a point P (corresponding to the Wallace point above), and the pedal planes of P for the 15 tetrahedra are coincident.—Steiner remarked, in 1845, that S is the locus of the centers of the hyperboloids for which, when their equations are in standard form,

$$(1/a^2) + (1/b^2) = 1/c^2,$$

and for which a given tetrahedron is self-polar.

Among many discussions of the surfaces, S, the following may be mentioned: by Geiser, in *Journal für die reine und angewandte Mathematik*, vol. 69, 1868, p. 199 f.; by F. E. Eckhardt, in *Mathematische Annalen*, vol. 5, 1872, pp. 30–49; by E. Janhke, *Archiv der Mathematik* (Grunert), third series, vol. 4, 1903, especially pp. 275–276; by J. Neuberg, *Archiv der Mathematik* (Grunert), third series, vol. 16, 1910, p. 18 f.; and by W. H. Salmon, in *Archiv der Mathematik* (Grunert), third series, vol. 18, 1911, pp. 154–164. ARC.

SOLUTIONS.

2719 [1918, 302]. Proposed by R. P. BAKER, University of Iowa.

Show that[2] $2x(\log x)^2 - x(x-1)(x+3)\log x + (x-1)^2(3x-1)$ is negative for $1 < x < \infty$.

SOLUTION BY OTTO DUNKEL, Washington University.

Denoting the given expression by $f(x)$, its first two derivatives may be written as follows:

$$f'(x) = 2(\log x)^2 - (x-1)(3x+7)\log x + 8(x-1)^2,$$

$$f''(x) = \frac{3x^2+2x-2}{x}\varphi(x), \qquad \varphi(x) = \frac{(13x-7)(x-1)}{3x^2+2x-2} - 2\log x.$$

It will be observed that $f(1) = f'(1) = f''(1) = 0$. It will be shown that $f''(x)$ is negative for $x > 1$ and it will then follow that $f'(x)$ is also negative, and hence $f(x)$ is likewise negative for $x > 1$. The derivative of $\varphi(x)$ may be written

$$\varphi'(x) = -\frac{2(x-1)^3(9x-4)}{x(3x^2+2x-2)^2}$$

and it is clearly negative for $x > 1$. Since $\varphi(1) = 0$, it follows that $\varphi(x) < 0$. The first factor of $f''(x)$, which may be denoted by $\psi(x)$, has the roots $-(\sqrt{7}+1)/3$ and $x_1 = (\sqrt{7}-1)/3 = .549$, and hence $\psi(x) > 0$ for $x > 1$. Therefore, $f''(x) < 0$ for $x > 1$ and the desired result is proved.

The form of $\varphi'(x)$ shows that the first of the derivatives of $f(x)$ which do not vanish for $x = 1$ is $f^{\text{vi}}(x)$. The expressions above give at once $f^{\text{vi}}(1) = -20$ and this shows that $f(x)$ is also negative for values of $x < 1$. It will now be shown that the interval for such values extends down to $x = 0$. In the interval $0 \leqq x < 1$ the derivative $\varphi'(x)$ vanishes only for 4/9 and it is negative before and positive after this value. Hence, $\varphi(x)$ has a minimum at this point. It will be found that $\varphi(4/9) = .312$, and so $\varphi(x)$ is positive from 0 to x_1. It is negative from x_1 to 1, since it vanishes at $x = 1$ and $\varphi'(x)$ is positive at all other points of this interval. On the other hand $\psi(x)$ is negative from 0 to x_1 and positive from x_1 to 1. At x_1 the product $\psi(x)\varphi(x)$ has a finite negative value and hence $f''(x)$ is negative from 0 to 1. From this follows that $f'(x)$ is positive and $f(x)$ is negative in the interval $0 \leqq x < 1$. Thus $f(x)$ is negative at every point of the interval $0 \leqq x < \infty$ except at the point $x = 1$ where it vanishes.

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[1] In *Annales de Mathématiques* (Gergonne), April, 1814, vol. 4, p. 320, the following problem was proposed (by Gergonne?) for solution: "The feet of the perpendiculars dropped on the faces of a tetrahedron from a point on the circumscribed sphere are coplanar." The incorrectness of this statement was proved by Durrande, in the same periodical, February, 1817, vol. 7, p. 255.

[2] It should be noted that this relation holds for "Naperian" but *not* for common logarithms. —EDITORS.